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The Kernel of the Composition of Two Operators

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1. INTRODUCTION

Let $A, B \in B(\mathscr{H})$, the algebra of bounded linear operators on the Hilbert space $\mathscr{H}$. This note gives conditions on A and B which are sufficient to imply that

$$\dim \ker AB \geqslant \dim \ker A. \tag{1.1}$$

(These conditions are stated explicitly in (3.1), (3.4), (3.8), and (3.9).) The following example shows that some restriction of the pair A, B is necessary.

EXAMPLE 1.1. Let $B \in B(\mathscr{H})$ be one-to-one but not onto. Let $\mathscr{K} \neq \{0\}$ be a closed subspace of $\mathscr{H}$ such that $\mathscr{K} \cap B\mathscr{H} = \{0\}$. Then (1.1) fails for every $A \in B(\mathscr{H})$ with kernel $\mathscr{K}$.

Although we have not characterized the pairs (A, B) for which (1.1) holds, an obvious desideratum, we have found the set of B such that (1.1) holds for every A in $B(\mathscr{H})$ (cf. 3.1) and the A's such that (1.1) holds for every B (cf. 3.9). In other words, while we have not characterized the relation defined by (1.1), we have found the horizontal and vertical lines it contains.

The usefulness of (1.1) may be illustrated by its application in [1]. However, there it is stated as though it were true for all $A, B \in B(\mathscr{H})$. One aim of this paper is to justify the applications of (1.1) actually made in [1].

Perhaps the most surprising feature of this note is its involvement with features of non-closed operator ranges. In Lemma 2.5 we construct a closed space of maximal dimension which meets the range of a given operator in $\{0\}$.

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We thank Chandler Davis for discussing these matters with us and guiding us to Ref. [2].

2. Some Useful Lemmas

Our first lemma simply lists some routine facts we will need.

Lemma 2.1. *Let $A, B \in B(\mathscr{H})$ and let $\mathscr{K}$, $\mathscr{L}$ be closed subspaces of $\mathscr{H}$. Then*

$$\dim \ker AB = \dim \ker B + \dim(\ker A \cap B\mathscr{H}). \tag{2.2}$$

$$\dim[\mathscr{K} \ominus (\mathscr{K} \cap \mathscr{L})] \leqslant \dim \mathscr{L}^{\perp}. \tag{2.3}$$

$$\dim \ker A + \dim A\mathscr{H} = \dim \mathscr{H}. \tag{2.4}$$

Proof of (2.3). When restricted to $\mathscr{K}$ the orthogonal projection of $\mathscr{H}$ onto $\mathscr{L}^{\perp}$ has kernel $\mathscr{K} \cap \mathscr{L}$. Hence it is a $1-1$ transformation of $\mathscr{K} \ominus (\mathscr{K} \cap \mathscr{L})$ into $\mathscr{L}^{\perp}$. ∎

Our second lemma is more interesting. Equation (2.2) shows that how a closed subspace of $\mathscr{H}$ and the range of an operator are positioned is germane to the problem we are considering. The following lemma provides the decisive information about such positioning, as Theorem 3.1 shows.

Lemma 2.5. *Let $B \in B(\mathscr{H})$. Then there exist closed subspaces $\mathscr{K}$, $\mathscr{L}$ such that*

$$\mathscr{K} \subset \mathscr{L}^{\perp}, \tag{2.6}$$

$$\dim \mathscr{K} = \dim \mathscr{L}^{\perp}, \tag{2.7}$$

$$\mathscr{K} \cap B\mathscr{H} = \{0\}, \tag{2.8}$$

and

$$\mathscr{L} \subset B\mathscr{H}. \tag{2.9}$$

Proof. Let $\{\mathscr{H}_g : g \in G\}$ be a family of separable, orthogonal, closed subspaces of $\mathscr{H}$ which reduce B. One such family is $\{\{0\}\}$. A well-known argument using Zorn's lemma shows that if the set of such families is ordered by inclusion it will contain a maximal element, say $\{\mathscr{H}_f \mid f \in F\}$, and the closed linear span $\mathscr{M}$ of the subspaces which are its members will be $\mathscr{H}$. Were $\mathscr{M} \neq \mathscr{H}$ the maximality could be contradicted by showing that if the closed subspace generated by applying to any fixed $0 \neq x \in \mathscr{M}^{\perp}$ the elements of the algebra generated by I, B, and B^* were added to the family, a larger family of the required type would result.

When $B\mathscr{H}_f$ is closed, we denote it $\mathscr{L}_f$ and we set $\mathscr{K}_f = \mathscr{H}_f \ominus B\mathscr{H}_f$. When $B\mathscr{H}_f$ is not closed, we set $\mathscr{L}_f = \{0\}$ and let $\mathscr{K}_f$ denote any closed infinite dimensional subspace of $\mathscr{H}_f$ which meets $B\mathscr{H}_f$ only in $\{0\}$. That such $\mathscr{K}_f$ exist follows from statements (a) and (b) of [2, p. 273]. (Note: Our $B\mathscr{H}_f$ corresponds to the space $\mathscr{R}$ in (b) and then $\mathscr{K}_f$ is the image under the unitary operator $W^{-2}V$ of the closed infinite dimensional subspace mentioned in (a).)

Let $\mathscr{K}$ denote the closed linear span of $\{\mathscr{K}_f : f \in F\}$ and $\mathscr{L}$ the closed linear span of $\{\mathscr{L}_f : f \in F\}$. Subscripting each $\mathscr{H}$, $\mathscr{K}$, and $\mathscr{L}$ in (2.6)–(2.9) with f gives four statements whose validity is clear for every $f \in F$ and from which (2.6)–(2.9) readily follow. ∎

3. The Main Results

First we consider (1.1) when B is fixed.

Theorem 3.1. *Let $B \in B(\mathscr{H})$.*
Then (1.1) *holds for every $A \in B(\mathscr{H}) \Leftrightarrow$*

$$\dim \mathscr{K} \leqslant \dim \ker B$$
$$\textit{for every closed subspace } \mathscr{K} \subseteq \mathscr{H} \textit{ such that } \mathscr{K} \cap B\mathscr{H} = \{0\}. \tag{3.2}$$

Proof. ($\Rightarrow$) If not, there exists a closed $\mathscr{K}$ such that $\mathscr{K} \cap B\mathscr{H} = \{0\}$ and $\dim \mathscr{K} > \dim \ker B$. Let A have kernel $\mathscr{K}$. Then $\dim \ker A > \dim \ker B = \dim \ker AB$.

($\Leftarrow$) Let $\mathscr{K}$ and $\mathscr{L}$ be as in Lemma 2.5. Then by hypothesis, $\dim \mathscr{K} \leqslant \dim \ker B$; and so by (2.3) and (2.7),

$$\dim[\ker A \ominus (\ker A \cap \mathscr{L})] \leqslant \dim \mathscr{L}^{\perp} = \dim \mathscr{K} \leqslant \dim \ker B. \tag{3.3}$$

Since

$$\ker A = (\ker A \cap \mathscr{L}) \oplus [\ker A \ominus (\ker A \cap \mathscr{L})],$$

it follows that

$$\dim \ker A \leqslant \dim(\ker A \cap B\mathscr{H}) + \dim \ker B = \ker AB$$

by (2.9), (3.3), and (2.2). ∎

Corollary 3.4. *Let $B \in B(\mathscr{H})$ have closed range.*
Then (1.1) *holds for every $A \in B(\mathscr{H})$ if and only if*

$$\dim (B\mathscr{H})^{\perp} \leqslant \dim \ker B. \tag{3.5}$$

Proof. Setting $\mathscr{K} = (B\mathscr{H})^{\perp}$ shows that (3.2) implies (3.5). Conversely, suppose $\mathscr{K}$ is closed and $\mathscr{K} \cap B\mathscr{H} = \{0\}$. Since $B\mathscr{H}$ is closed it can play the role of $\mathscr{L}$ in (2.3). Hence by (2.3) and (3.5),

$$\dim \mathscr{K} \leqslant \dim(B\mathscr{H})^{\perp} \leqslant \dim \ker B. \quad \blacksquare$$

Remark 3.6. We actually showed that (3.5) is necessary even when $B\mathscr{H}$ is not closed.

Remark 3.7. Corollary 3.4 justifies the use of (1.1) made in [1, p. 403, line 8] since the role of B there is played by an orthogonal projection.

COROLLARY 3.8. *The inequality* (1.1) *holds for all* $A, B \in B(\mathscr{H})$ *if and only if* $\dim \mathscr{H} < \infty$.

Proof. The example given in Section 1 shows that $\dim \mathscr{H} < \infty$ is necessary even if (1.1) is to hold for all $A, B \in B(\mathscr{H})$ with $B\mathscr{H}$ closed. Conversely, if $\dim \mathscr{H} < \infty$ then (2.4) and subtraction (which can not be justified when $\dim \mathscr{H} = \infty$) show that equality holds in (3.5). $\blacksquare$

Next we consider the case where A is fixed.

THEOREM 3.9. *Let* $A \in B(\mathscr{H})$.
Then $\dim \ker AB \geqslant \dim \ker A$ *for every* $B \in B(\mathscr{H})$,

$$\Leftrightarrow \quad \text{(a)} \ \ker A = \{0\}$$

or

$$\text{(b)} \ \dim A\mathscr{H} < \dim \mathscr{H}.$$

Remark. $\dim A\mathscr{H} = \dim(\ker A)^{\perp}$.

Proof. Suppose the inequality holds for every $B \in B(\mathscr{H})$. To prove that (a) or (b) must hold we show that if (b) fails, (a) must hold. If (b) fails, $\dim(\ker A)^{\perp} = \dim A\mathscr{H} = \dim \mathscr{H}$; so there exists an isometry $B \in B(\mathscr{H})$ of $\mathscr{H}$ onto $(\ker A)^{\perp}$. Then

$$\dim \ker A \leqslant \dim \ker AB = \{0\}.$$

Conversely, if (a) holds, the inequality is trivial for every B. Suppose (b) holds and let $B \in B(\mathscr{H})$. Let P be the orthogonal projection with kernel $\ker A$ and range $(\ker A)^{\perp}$. Then by 2.4

$$\begin{aligned} \dim B\mathscr{H} &= \dim \ker(P|_{B\mathscr{H}}) + \dim(PB\mathscr{H}) \\ &= \dim(\ker A \cap B\mathscr{H}) + \dim(PB\mathscr{H}) \\ &\leqslant \dim(\ker A \cap B\mathscr{H}) + \dim(\ker A)^{\perp}. \end{aligned} \tag{3.10}$$

If $\dim \ker B = \dim \mathscr{H}$ the validity of the inequality is clear; so assume $\dim \ker B < \dim \mathscr{H}$. From (2.4) it follows that $\dim B\mathscr{H} = \dim \mathscr{H}$ (except in the case that $\dim \mathscr{H} < \infty$, where Corollary 3.8 tells us that (1.1) holds for all $A, B \in B(\mathscr{H})$). Then, since

$$\dim(\ker A)^{\perp} = \dim A\mathscr{H} < \dim \mathscr{H},$$

we can conclude from (3.10) that $\dim(\ker A \cap B\mathscr{H}) = \dim \mathscr{H}$. Then (2.2) completes the proof. ∎

Remark 3.11. Equation (2.2) shows that whether (1.1) is valid for A, B depends only on $\ker A$, $\ker B$, $B\mathscr{H}$, and their position in $\mathscr{H}$.

Remark 3.12. The question of when (1.1) holds is more naturally a question of vector spaces and linear transformations than a question about Hilbert spaces. For example, (1.1) is valid for all linear operators over a finite dimensional space although Corollary 3.8 seems to suggest that Hilbert space structure plays some role. It has been convenient to confine the analysis to Hilbert spaces because the necessary information about nonclosed operator ranges is available for them [2]. We expect that our theorems will be readily extendable to other spaces where the structure of certain of their basic features, such as nonclosed operator ranges is sufficiently clear.

References

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2. P. A. Filmore and J. P. Williams, On operator ranges, *Adv. in Math.* **7** (1971), 254–281.